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XXI. *On the Illustration of the Properties of the Electric Field by Means of Tubes of Electrostatic Induction.* *By* J. J. THOMSON, *M.A.*, *F.R.S.*, *Cavendish Professor of Experimental Physics, Cambridge* *.

I HAVE attempted in the following pages to develop a method of expressing the various processes which occur in the electric field in terms of changes in the form or position of tubes of electrostatic induction which are assumed to be distributed throughout the field, in the hope that it may help the student to obtain a physical interpretation of results which are perhaps too frequently regarded as entirely expressed by equations. Methods such as this, of materializing, as it were, mathematical conceptions, seem to have a use even where, as in the case of Electricity, the analytical theory is well established; for any method which enables us to form a mental picture of what goes on in the electric field has a freshness and a power of rapidly giving the main features of a phenomenon, as distinct from the details, which few can hope to derive from purely analytical methods. Experience has, I think, shown that Maxwell's conception of electric displacement is of somewhat too general a character to lend itself easily to the formation of a conception of a mechanism which would illustrate by its working the processes going on in the electric field. For this purpose the conception of tubes of electrostatic induction introduced by Faraday seems to possess many advantages. If we regard these tubes as having a real physical existence, we may, as I shall endeavour

* Communicated by the Author.

to show, explain the various electrical processes,—such as the passage of electricity through metals, liquids, or gases, the production of a current, magnetic force, the induction of currents, and so on,—as arising from the contraction or elongation of such tubes and their motion through the electric field.

We might, as we shall see, have taken the tubes of magnetic force as the quantity by which to express all the changes in the electric field; the reason I have chosen the tubes of electrostatic induction is that the intimate relation between electrical charges and atomic structure seems to point to the conclusion that it is the tubes of electrostatic induction which are most directly involved in the many cases in which electrical charges are accompanied by chemical ones.

We may regard the method from one point of view as being a kind of molecular theory of electricity, the properties of the electric field being explained as the effects produced by the motion of multitudes of tubes of electrostatic induction; just as in the molecular theory of gases the properties of the gas are explained as the result of the motion of its molecules.

As the principal reason for expressing the effects in terms of the tubes of electrostatic induction is the close connexion between electrical and chemical properties, we shall begin by considering at some length the connexion between these tubes of electrostatic induction and the atoms of bodies.

We assume, then, that the electric field is full of tubes of electrostatic induction, that these are all of the same strength, and that this strength is such that when a tube falls on a conductor it corresponds to a negative charge on the conductor equal in amount to the charge which in electrolysis we find associated with an atom of a univalent element.

These tubes must either form closed circuits or they must end on atoms, any unclosed tube being a tube connecting two atoms. In this respect the tubes resemble lines of vorticity in hydrodynamics, as these lines must either be closed, or have their extremities on a boundary of the fluid.

We may suppose that associated with these tubes of electrostatic induction there is a distribution of velocity, both in themselves and in the surrounding æther, and that the energy due to this motion of the medium constitutes the energy which is distributed throughout the electric field. In addition to there being this energy in the medium, the incidence of a tube of force on an atom may modify the internal motion of the atom, and thus alter its energy, so that, in addition to the energy in the field, there may be a certain amount of energy due to the alteration in the motion of the atoms: this may be represented by the addition to the ordinary expression for the energy of a term w for each atom on which a unit tube falls, $-w$ for each atom which a tube leaves; we shall suppose that w depends on the nature of the atom, that it is not the same for zinc as for copper and so on. The existence of this energy will produce the same effect as if the atoms of different substances attracted electricity with different degrees of intensity: this is the assumption made by von Helmholtz, and it has been shown by him to be sufficient to account for contact electricity.

The ends of an unclosed tube of induction are places where electrification exists, and therefore are always situated on matter. According to our view, the ends of a tube of finite length are on free atoms as distinct from molecules, the atoms in the molecule being connected by a short tube whose length is of the order of the molecular distance. On this view, therefore, the existence of free electricity, whether on a metal, an electrolyte, or a gas, always denotes the existence of free atoms. The production of electrification must be accompanied by chemical dissociation, the disappearance of it by chemical combination; changes in electrification are on this view always accompanied by chemical changes. This was long thought to be a peculiarity of the passage of electricity through electrolytes, but recent experiments seem to show that it is also the case when electricity passes through gases. Thus, for example, those gases which conduct readily when hot are those which dissociate when heated, and are thus undergoing chemical changes when the electricity passes through them. Again, it is known that the passage of electricity through many gases causes chemical changes to take place—the production of ozone is the most familiar instance of this, but there are a multitude of others. Lastly, R. v. Helmholtz and Richarz have found that when electricity passes through a gas, a steam-jet in the neighbourhood is influenced in the same way as it is when free atoms are produced by chemical changes. All these results seem to point to the conclusion that the passage of electricity through gases is accompanied by changes in the pairing of the atoms of the gas. Although we have no such direct evidence of the same effect when electricity passes through metals, it must be borne in mind that direct evidence in this case is very much more difficult to obtain, and there are many reasons for taking the view that the passage of electricity through metals is performed in much the same way as it is through electrolytes and gases.

We will begin by considering how metallic conduction differs from electrolytic. In the first place, as the temperature increases the conductivity of electrolytes, as a general rule, increases, while that of metals diminishes. This rule is not, however, without exceptions: there are cases in which, though the conduction is not usually supposed to be electrolytic, the conductivity increases as the temperature increases. Carbon is a striking instance of this, and quite lately Feussner has prepared alloys of manganese and copper whose conductivities show the same peculiarity. These exceptions are sufficient to show that increase of conductivity with the temperature is not a sufficient test to separate electrolytic from metallic conduction.

If we regard the passage of electricity through a body as essentially bound up with chemical changes, it does not seem surprising that an increase in temperature may produce opposite effects on the conductivities of two substances, even though in both cases the conduction was effected by changes in the pairings of the atoms. For the action of an increase of temperature has a two-fold effect on the processes which, on this view, accompany electric conduction. In the first place, it may promote the splitting up of the molecules into atoms which, on this theory, forms one part of the process of conduction; but, on the other hand, after the molecules are split up it retards their reunion, which forms another part of the process. And, again, an increase in the temperature increases the distance between the molecules, and this will also retard the rate at which chemical interchange takes place. The fact that the metals are solids is no reason why the conductivity through them should not be electrolytic in its nature, for there are many instances of solid electrolytes; thus Lehmann has shown that electrolysis takes place through a crystal of silver iodide placed between silver electrodes without any change being perceptible in the shape or size of the crystal, though it was watched through a microscope whilst the current was passing.

With regard to the appearance of the products of chemical decomposition at the electrodes, we could not expect to get any evidence of this in the case of the elementary metals; the case of alloys seems more hopeful; but Professor Roberts Austen has examined several alloys through which a powerful current had been passed without detecting any difference in the composition of the alloy at the terminals. This result does not, however, seem to me to prove that the conduction was not electrolytic; for some alloys are little more than mixtures, whilst others behave as if they were solutions of one metal in another; and in neither of these cases could we expect to get any change in the composition of the alloy at the electrodes. We could only expect to find this when we used an alloy in which the connexion between the constituents could be regarded as of such a definite character that in the molecule of the alloy one metal could be regarded as the positive, the other as the negative element. The alloys used by Prof. Roberts Austen do not seem to have been of this character. The reasons which account for the absence of change in the constitution of the alloy at the electrodes will also account for the absence of polarization.

Though the electrical conductivities of the metals are enormously greater than those of electrolytes, there does not seem to be any abrupt change from the conductivity in cases where it is manifestly electrolytic, such as fused lead chloride, to those in which it is not recognized as being of that nature, as in carbon. The following table, giving the electric conductivity of some substances, will show this:—

Silver	63
Mercury	1
Gas-carbon	1×10^{-3}
Tellurium	4×10^{-4}
Fused lead chloride	2×10^{-4}

There is a greater disproportion between the thermal conductivities of silver and cement than there is between the electrical conductivities of mercury and fused lead chloride; but no one argues that, on this account, the method by which heat is propagated in silver is essentially different from that by which it is propagated in cement.

It is also suggestive that the substances which are intermediate in their chemical properties between the metals and the non-metals, such as phosphorus, selenium, and tellurium, possess properties with regard to metallic conduction intermediate between those of metals and electrolytes,—thus, as shown by W. Siemens, the resistance of some of the modifications of selenium increases with the temperature, while that of other modifications diminishes,—and that some of the modifications seem to show polar effects,—thus, when one electrode is large and the other small, the current is greater when the large electrode is negative than when it is positive. The changes in the chemical properties of the substance seem to proceed step by step with the changes in their behaviour with regard to electrical conduction.

If we accept the electromagnetic theory of light, we have an additional reason for supposing that the processes concerned in metallic conduction are the same as those in electrolytic. For the opacity of thin metal films is enormously less than that theory would indicate, if the conductivity of the film for the very rapid electrical vibrations which constitute light were the same as for steady currents. In this respect the metals behave very much like electrolytes, for these act as dielectrics to the light vibrations and as conductors for steady or oscillating currents, provided the period of vibration is very much greater than that of the light-vibrations. On the view we have taken of metallic conduction, since the process of dissociation and recombination takes a finite time, if the polarization is reversed in less than this time, the old polarization will not have had time to disappear before the new is superposed, and the metal will, under these circumstances, behave more like an insulator than a conductor.

We can easily find an expression for the time T taken by a tube of electrostatic induction to disappear (that is, to contract to a length comparable with that between the atoms of a molecule). Let E be the electromotive intensity at any point, K the specific inductive capacity of the medium; then the number of tubes of electrostatic induction passing through unit area is

$$\frac{K}{4\pi}E.$$

Since T is the time taken by one tube to disappear, the number of tubes which disappear in the conductor in unit time is $KE/4\pi T$; the number of tubes which disappear in unit time is equal, however, to the current c through unit area. Hence

$$c = \frac{KE}{4\pi T}.$$

Thus $4\pi T/K$ is the specific resistance σ of the conductor; hence, if $\{K\}$ be the electrostatic measure of K, we have

$$T = \frac{K}{4\pi} \frac{\sigma}{9 \times 10^{20}}.$$

The following table contains the values of T/K for a few substances:—

Silver	$1\cdot5 \times 10^{-19}$
Lead	$1\cdot8 \times 10^{-18}$
Water with 8·3 per cent. of H_2SO_4 .	$3\cdot1 \times 10^{-13}$

The value of K for anything like a good conductor has never been measured; but since substances which show the least trace of conductivity, such as water or alcohol, have specific inductive capacities ranging from 70 to 100, it is probable that for good conductors K is exceedingly large. If, however, its value for metals were no greater than that for distilled water, the time required for the disappearance of a tube of force would be comparable with the time of vibration of a light-wave; so that the conductivity would be much smaller for these waves than for steady currents.

We may picture to ourselves the tubes of electrostatic induction shortening in a conductor in some such way as the following:—Let us take the case of a condenser discharging through the gas separating its plates. Then, before discharge, we have a tube stretching from an atom O on the positive plate to another atom P on the negative one. The molecules AB, CD, . . . of the intervening gas will be polarized by the induction, the tubes of force connecting the atoms in these molecules pointing in the negative direction; as the strength of the field increases the tube in the molecule AB will lengthen and bend towards the tube OP, until when the field is sufficiently strong the molecular tube runs up into the tube OP. The tubes then break up into two tubes OA and PB, and the tube OA shortens to molecular dimensions. The result of this operation is that the tube PO has shortened to PB, and the atoms O and A have formed a molecule. This process is then continued from molecule to molecule until the tube PO has contracted to molecular dimensions. Instead of the tube PO jumping from molecule to molecule, several molecules may form a chain and be affected at once; in this case the tube would shorten by the length of the chain in the same time as on the previous hypothesis it shortened by the distance between two molecules.

The remarkable agreement between the values of the velocity of the ions as calculated by Hittorf and Kohlrausch and those found by Prof. Lodge in the experiments described in the British Association Report, 1887, makes it essential that any theory of conduction through electrolytes should lead to the same expressions for the ionic velocities. According to the preceding theory of conduction, if the molecules of the electrolyte form chains between the electrodes, the sum of the distances traversed by the anion and cation each time the tube of electrostatic induction breaks down is d, where d is the distance between two molecules of the electrolyte measured along a tube of induction. If the tube breaks down n times a second, the sum of the distances traversed by the ion and cation in one second is nd; so that, if u and v are the velocities of the anion and cation respectively, $u+v=nd.$

If ϵ is the charge on one of the atoms of the electrolyte, m the number of molecules per unit of area of a plane in the electrolyte parallel to the electrodes, then, when the molecules are polarized, the charge per unit area on the end of the chain of molecules is $m\epsilon$; this equals the surface-density on the electrodes $\mathrm{KE}/4\pi$, where E is the electromotive intensity and K the specific inductive capacity of the electrolyte. Hence

$$m\epsilon = \frac{\mathrm{K}}{4\pi}\mathrm{E}.$$

If c be the conductivity of the electrolyte, N the number of molecules in unit volume, we have

$$\frac{\mathrm{K}}{4\pi}n = c, \quad m = \mathrm{N}d.$$

Making these substitutions, we find

$$u + v = \frac{c\mathrm{E}}{\mathrm{N}\epsilon},$$

which is the same as the expression for the sum of the velocities given by Kohlrausch. The ratio of the velocities will follow exactly the same way from the migration-data whichever theory we adopt; so that there is nothing in Prof. Lodge's confirmation of Kohlrausch's expressions for the velocity of the ions inconsistent with the theory of conduction we are describing.

We will now leave the consideration of the behaviour of these tubes in conductors, and proceed to discuss their properties when moving through the dielectric.

Let f, g, h denote the number of unit tubes parallel to the axes of x, y, z respectively—in other words, the components of the electric displacement; and let us suppose that these tubes are moving with the velocity u, v, w parallel to the axes of coordinates.

Let us consider the increase in the number of tubes parallel to x which occurs in a time δt in an element of volume dx, dy, dz.

The increase due to the passage of the tubes across the faces of the element is

$$-\delta t\left(\frac{d}{dx}(fu) + \frac{d}{dy}(fv) + \frac{d}{dz}(fw)\right)dx\,dy\,dz.$$

The increase due to the deformation of the tubes inside the element is

$$\delta t\left(f\frac{du}{dx} + g\frac{du}{dy} + h\frac{du}{dz}\right)dx\,dy\,dz.$$

The rate of increase in the number of tubes parallel to x is thus

$$f\frac{du}{dx}+g\frac{du}{dy}+h\frac{du}{dz}-\left(\frac{d}{dx}(fu)+\frac{d}{dy}(fv)+\frac{d}{dz}(fw)\right);$$

which may be written as

$$\frac{d}{dy}(gu-fv)-\frac{d}{dz}(fw-hu)-u\left(\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}\right);$$

and

$$\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}=\rho,$$

ρ being the density of the free electricity.

Hence df/dt, the rate of increase in the number of tubes parallel to x, may be written as

$$\frac{df}{dt}=\frac{d}{dy}(gu-fv)-\frac{d}{dz}(wf-uh)-u\rho;$$

and similarly

$$\frac{dg}{dt}=\frac{d}{dz}(vh-wg)-\frac{d}{dx}(gu-fv)-v\rho,$$

$$\frac{dh}{dt}=\frac{d}{dx}(wf-uh)-\frac{d}{dy}(vh-wg)-w\rho.$$

But since $\frac{df}{dt}+u\rho$, $\frac{dg}{dt}+v\rho$, $\frac{dh}{dt}+w\rho$ are the components of the current parallel to x, y, z, we have, if α, β, γ are the components of the magnetic force,

$$4\pi\left(\frac{df}{dt}+u\rho\right)=\frac{d\gamma}{dy}-\frac{d\beta}{dz},$$

$$4\pi\left(\frac{dg}{dt}+v\rho\right)=\frac{d\alpha}{dz}-\frac{d\gamma}{dx},$$

$$4\pi\left(\frac{dh}{dt}+w\rho\right)=\frac{d\beta}{dx}-\frac{d\alpha}{dy}.$$

Hence we may regard the moving tubes of electrostatic induction as producing a magnetic force whose components α, β, γ parallel to the axes of x, y, z are given by

$$\left.\begin{aligned}\alpha &= 4\pi(hv-gw),\\ \beta &= 4\pi(fw-hu),\\ \gamma &= 4\pi(gu-fv).\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In other words, a moving tube of electrostatic induction may be regarded as producing a magnetic force at right angles both to itself and the direction in which it is moving, and whose magnitude is 4π times the strength of the tube multiplied by its velocity at right angles to its direction. The direction of the force is such that the magnetic force and rotation from the direction of motion to that of the tube are related like translation and rotation in a right-angled screw.

Let us first consider the case where all the tubes are moving with the same velocity in a field whose magnetic permeability is unity.

The energy in the magnetic field per unit volume is

$$\frac{1}{8\pi}(\alpha^2+\beta^2+\gamma^2),$$

or substituting for α, β, γ their values from (1),

$$2\pi\left\{(hv-gw)^2+(fw-hu)^2+(gu-fv)^2\right\}.$$

The momentum U per unit volume parallel to x is the differential coefficient of this expression with respect to u, *i. e.*

$$4\pi\left\{g(gu-fv)-h(fw-hu)\right\},$$

or

$$U=g\gamma-h\beta.$$

Similarly, if V, W are the components of the momentum parallel to y and z, we have

$$\left.\begin{aligned} V&=h\alpha-f\gamma,\\ W&=f\beta-g\alpha.\end{aligned}\right\} \quad \ldots\ldots \quad (2)$$

Thus the momentum per unit volume possessed by the moving tube is at right angles to the tube and to the magnetic force produced by it, and equals the product of the strength of the tube and the magnetic force.

The electromotive intensity parallel to x produced by the moving tube can be got by differentiating the expression for the Kinetic Energy with respect to f; in this way we obtain the following expressions for X, Y, Z, the components of the electromotive intensity:—

$$\left.\begin{aligned} X&=w\beta-v\gamma,\\ Y&=u\gamma-w\alpha,\\ Z&=v\alpha-u\beta.\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (3)$$

Thus the electromotive intensity produced by the motion of the tube is equal to the product of the velocity of the tube and the magnetic force produced by it, and is at right angles to both the direction of motion of the tube and the magnetic force produced by it.

From equation (3) we see that

$$\frac{dZ}{dy}-\frac{dY}{dz}=v\frac{d\alpha}{dy}+w\frac{d\alpha}{dz}-u\left(\frac{d\beta}{dy}+\frac{d\gamma}{dz}\right)+\alpha\left(\frac{dv}{dy}+\frac{dw}{dz}\right)-\beta\frac{du}{dy}-\gamma\frac{du}{dz};$$

or, since

$$\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}=0,$$

we have

$$\frac{dZ}{dy}-\frac{dY}{dz}=\frac{d}{dx}(u\alpha)+\frac{d}{dy}(v\alpha)+\frac{d}{dz}(w\alpha)-\alpha\frac{du}{dx}-\beta\frac{du}{dy}-\gamma\frac{du}{dz}. \qquad (4)$$

The right-hand side of this equation is equal to $-\frac{d\alpha}{dt}$, the rate of diminution in the number of tubes of magnetic force parallel to the axis of x.

Hence, since

$$\int(Xdx+Ydy+Zdz)$$

taken round a closed circuit is equal to

$$\iint\left\{l\left(\frac{dZ}{dy}-\frac{dY}{dz}\right)+m\left(\frac{dX}{dz}-\frac{dZ}{dx}\right)+n\left(\frac{dY}{dx}-\frac{dX}{dy}\right)\right\}dS,$$

taken over any surface entirely surrounded by the circuit, if l, m, n are the direction-cosines of the normal to the surface, we see by (4) that the line-integral of the electromotive force taken round a closed circuit is equal to the rate of diminution of the number of lines of magnetic force passing through the circuit.

Collecting these results, we see that a tube of electrostatic induction when in motion produces (1) a magnetic force at right angles to the tube and the direction of motion, (2) a momentum at right angles to the tube and the magnetic force produced by it, (3) an electromotive intensity at right angles to the direction of motion of the tube and the magnetic force produced by it.

The momentum and the electromotive intensity are thus both in the plane containing the direction of the tube and its velocity; the first of these is at right angles to the tube, the second to the velocity.

We have hitherto only considered the case of one tube, or rather of a set of tubes, moving with a common velocity. We can, however, without difficulty extend these results to the case where we have any number of tubes moving with any velocities.

Let us suppose we have the tubes f_1, g_1, h_1 moving with the velocities u_1, v_1, w_1, the tubes f_2, g_2, h_2 moving with the velocities u_2, v_2, w_2, and so on. Then the rate of increase in the number of tubes in the element is

$$\Sigma\left(f\frac{du}{dx}+g\frac{du}{dy}+h\frac{du}{dz}\right)-\Sigma\left(\frac{d}{dx}(fu)+\frac{d}{dy}(fv)+\frac{d}{dz}(fw)\right).$$

This may be written as

$$\frac{d}{dy}\Sigma(gu-fv)-\frac{d}{dz}\Sigma(fw-hu)-\Sigma u\rho.$$

Hence we see, as before, that the collection of tubes may be regarded as producing a magnetic force whose components α, β, γ are given by

$$\left.\begin{aligned}\alpha &= 4\pi\Sigma(hv-gw),\\ \beta &= 4\pi\Sigma(fw-hu),\\ \gamma &= 4\pi\Sigma(gu-fv).\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad . \quad (5)$$

The kinetic energy T per unit volume due to the motion of these tubes is

$$\frac{1}{8\pi}(\alpha^2+\beta^2+\gamma^2)$$

or

$$2\pi\left[\{\Sigma(hv-gw)\}^2+\{\Sigma(fw-hu)\}^2+\{\Sigma(gu-fv)\}^2\right].$$

Thus the momentum parallel to x of the tube with suffix (1) $d\mathrm{T}/du_1$

$$\begin{aligned}&= 4\pi\{g_1\Sigma(gu-fv)-h_1\Sigma(fw-hu)\}\\ &= g_1\gamma-h_1\beta.\end{aligned}$$

Thus the components U, V, W of the momenta parallel to the axes of x, y, z respectively are given by the equations

$$\left.\begin{aligned}\mathrm{U} &= \gamma\Sigma g-\beta\Sigma h,\\ \mathrm{V} &= \alpha\Sigma h-\gamma\Sigma f,\\ \mathrm{W} &= \beta\Sigma f-\alpha\Sigma g.\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad . \quad (6)$$

Thus, when we have a number of tubes moving about, the resultant momentum at any point is perpendicular to both the resultant magnetic force and the resultant electric displacement, and is equal to the product of these two quantities into the sine of the angle between them.

The electromotive intensity X parallel to the axis of x is equal to the mean of $d\mathrm{T}/df$.

Hence

$$\left.\begin{aligned} \mathrm{X} &= 4\pi\left(\overline{w\Sigma(fw-hu)-v\Sigma(gu-fv)}\right) \\ &= \left(\overline{w\beta-v\gamma}\right) \\ &= \beta\bar{w}-\gamma\bar{v}. \end{aligned}\right\} \quad . \quad . \quad (7)$$

Similarly, if Y and Z are the components parallel to the axes of y and z, we have

$$\mathrm{Y}=\gamma\bar{u}-\alpha\bar{w},$$
$$\mathrm{Z}=\alpha\bar{v}-\beta\bar{u},$$

where a bar placed over any quantity indicates that the mean value of the quantity is to be taken.

Thus, when a system of tubes of electrostatic induction is in motion, the electromotive intensity is at right angles both to the resultant magnetic force and the mean velocity, and is equal to the product of these two quantities into the sine of the angle between them.

From the preceding equations, we see that if we have a great number of positive and negative tubes moving about, if the positive tubes move in one direction and the negative ones in the opposite, there will be a resultant magnetic force, but if there are as many positive as negative tubes in each unit of volume, there will be no resultant momentum ; and if there are as many moving in one direction as the opposite, there will be no resultant electromotive intensity, due to the motion of these tubes. We see then that when the electromagnetic field is in a steady state, the motion of the tubes of electrostatic induction in the field will be a kind of shearing of the positive past the negative tubes, the positive tubes moving at one direction, and the negative at an equal rate in the opposite. When, however, the field is not in a steady state, this ceases to be the case, and then the electromotive forces due to induction are developed.

We get from equation (7),

$$\frac{d\mathrm{Z}}{dy}-\frac{d\mathrm{Y}}{dz}=\frac{d}{dx}(u\alpha)+\frac{d}{dy}(v\alpha)+\frac{d}{dz}(w\alpha)-\left(\alpha\frac{d\bar{u}}{dx}+\beta\frac{d\bar{u}}{dy}+\gamma\frac{d\bar{u}}{dz}\right),$$

since

$$\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}=0.$$

If we suppose the magnetic lines of force to be moving with the velocities $\bar{u}$, $\bar{v}$, $\bar{w}$, the right-hand side of this equation represents the rate of diminution of α.

When the tubes of electrostatic induction enter a conductor, their ends, as we have seen, get attached to atoms of the conductor, and the momentum of the tube gets transferred to the conductor. Let us consider a small portion of a conductor conveying a current, the area of the portion being so small that we may consider the magnetic force over it to be constant. The momentum parallel to x of a tube entering it is

$$g\gamma - h\beta\,;$$

thus the momentum parallel to x which enters the element in unit time is

$$\iint \Sigma\{(g\gamma - h\beta)(lu + mv + nw)d\mathrm{S},$$

where $d\mathrm{S}$ is an element of the surface of the element, and l, m, n the direction-cosines of the normal to this surface. The above expression may be written in the form

$$\gamma\iint \Sigma\{\, g\,(lu + mv + nw)\}\, d\mathrm{S} - \beta\iint \Sigma\{h(lu + mv + nw)\}d\mathrm{S}.$$

Now

$$\iint \Sigma\{g(lu + mv + nw)d\mathrm{S},$$

and

$$\iint \Sigma\{h(lu + mv + nw)d\mathrm{S}$$

are the number of tubes of force parallel to x and y respectively which enter the element in unit time, that is, they are the components q and r of the current parallel to y and z respectively. Thus the momentum parallel to x communicated in unit time to the conductor, in other words, the force parallel to x acting on the conductor, is equal to

$$\gamma q - \beta r.$$

Similarly, the forces parallel to y and z are respectively

$$\alpha r - \gamma p,$$
$$\beta p - \alpha q.$$

These are the ordinary expressions for the force acting on a conductor carrying a current in a magnetic field.

When, as in the above investigation, we regard the force on a conductor carrying a current as due to the tubes of electrostatic induction which enter the circuit giving up their momentum to it, the origin of the force between two currents will be very much the same as that of the attraction between two bodies on Le Sage's theory of gravitation. Thus, for example, let us take the case of two straight parallel currents, A and B, flowing in the same direction, and let us suppose that A is to the left of B ; then more tubes of force will enter A from the left than from the right, because some of those which would have come from the right if B had been absent will be absorbed by B; thus in unit time more momentum having the direction of left to right will enter A than that having the opposite direction ; thus A will move towards the right, that is, towards B, while for a similar reason B will move towards A.

We have now shown that we can explain the properties of the electromagnetic field if we suppose that throughout that field tubes of electrostatic induction in rapid motion are distributed, and that we can obtain the ordinary equations of the electromagnetic field if we start with the principle that the line-integral of the magnetic force round a closed curve is equal to the rate of increase of the number of tubes of electrostatic induction passing through that curve.

We shall now proceed to discuss some special problems by the light of this theory. The first we shall take is that of a sphere charged with electricity, and moving with the velocity w parallel to the axis of z. When things have reached a steady state, we may suppose that the sphere and the tubes of electrostatic induction emanating from it move like a solid body: we shall now consider the effect produced by pushing these tubes of force through the æther. Thus, if α, β, γ are the components of the magnetic force at any point, f, g, h those of the electric displacement at the same point, we have, by equation (1),

$$\alpha = -4\pi w g,$$
$$\beta = 4\pi w f,$$
$$\gamma = 0.$$

If X_1, Y_1, Z_1 are the components of the electromotive intensities due to the motion of the tubes of electrostatic induction, we have, by equation (3),

$$X_1 = w\beta,$$
$$Y_1 = -w\alpha,$$
$$Z_1 = 0.$$

The total electromotive intensity, whose components are X, Y, Z, will be given by the equations

$$X = \quad w\beta - \frac{d\psi}{dx},$$

$$Y = -w\alpha - \frac{d\psi}{dy},$$

$$Z = \qquad - \frac{d\psi}{dz};$$

where ψ is a function which we must proceed to determine.

Substituting the above values for α and β, and remembering that

$$X=\frac{4\pi}{K}f, \quad Y=\frac{4\pi}{K}g, \quad Z=\frac{4\pi}{K}h,$$

we get

$$\frac{4\pi}{K}f=4\pi w^2 f-\frac{d\psi}{dx},$$

$$\frac{4\pi}{K}g=4\pi w^2 g-\frac{d\psi}{dy},$$

$$\frac{4\pi}{K}h= \qquad -\frac{d\psi}{dz}.$$

Putting $1/K=V^2$, where V is the velocity of light through the dielectric, and remembering that

$$\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}=0.$$

we get

$$\frac{d^2\psi}{dx^2}+\frac{d^2\psi}{dy^2}+\left(1-\frac{w^2}{V^2}\right)\frac{d^2\psi}{dz^2}=0,$$

which, if

$$z_1=\frac{z}{\left\{1-\frac{w^2}{V^2}\right\}^{\frac{1}{2}}},$$

may be written as

$$\frac{d^2\psi}{dx^2}+\frac{d^2\psi}{dy^2}+\frac{d^2\psi}{dz_1^2}=0.$$

The solution of which is

$$\psi=\frac{A}{\{x^2+y^2+z_1^2\}^{\frac{1}{2}}}.$$

Since

$$(V^2-w^2)f=-\frac{1}{4\pi}\frac{d\psi}{dx},$$

$$(V^2-w^2)g=-\frac{1}{4\pi}\frac{d\psi}{dy},$$

$$V^2 h=-\frac{1}{4\pi}\frac{d\psi}{dz}.$$

And since

$$\iint\left(f\frac{x}{a}+g\frac{y}{a}+h\frac{z}{a}\right)dS=e,$$

when the integration is extended over a sphere of radius a concentric with the moving sphere, and e is the charge on that sphere, substituting the values of f, g, h in this equation in terms of ψ we find, if $w/V<1$,

$$A=eV\sqrt{V^2-w^2}.$$

Thus the electrostatic potential ϕ is given by the equation

$$\psi=\frac{e\left\{1-\frac{w^2}{V^2}\right\}^{\frac{1}{2}}V^2}{\left(x^2+y^2+\frac{z^2}{1-\frac{w^2}{V^2}}\right)^{\frac{1}{2}}}.$$

Thus the tubes of induction are no longer uniformly distributed, but are, in consequence of the electrical inertia, crowded towards the region in proximity to the equatorial plane of the sphere, where the displacements do not vary so quickly as along the axis.

The displacements are radial and are given by

$$\frac{f}{x}=\frac{g}{y}=\frac{h}{z}=\frac{e}{4\pi\left\{1-\frac{w^2}{V^2}\right\}^{\frac{1}{2}}}\frac{1}{\left(x^2+y^2+\frac{z^2}{1-\frac{w^2}{V^2}}\right)^{\frac{3}{2}}}.$$

The components a and β of the magnetic force are given by

$$a=-\frac{ewy}{\left\{1-\frac{w^2}{V^2}\right\}}\frac{1}{\left(x^2+y^2+\frac{z^2}{1-\frac{w^2}{V^2}}\right)^{\frac{3}{2}}},$$

$$\beta=\frac{ewx}{\left(1-\frac{w^2}{V^2}\right)^{\frac{1}{2}}}\frac{1}{\left(x^2+y^2+\frac{z^2}{1-\frac{w^2}{V^2}}\right)^{\frac{3}{2}}}.$$

(See Heaviside, Phil. Mag. April 1889; J. J. Thomson, Phil. Mag. July 1889.)

The momentum in the medium is at right angles to the displacement and the magnetic force, and is therefore at right angles to the radius in the plane containing the radius and the direction of motion; its magnitude per unit volume is by equation (2),

$$\frac{w}{4\pi}\frac{e^2}{1-\frac{w^2}{V^2}}\cdot\frac{x^2+y^2}{\left(x^2+y^2+\frac{z^2}{1-\frac{w^2}{V^2}}\right)^3}$$

Let us now suppose that the sphere is moving in a uniform field of magnetic force, where the components of the external magnetic force are $\alpha_0, \beta_0, \gamma_0$, the momentum per unit volume parallel to x is

$$g\gamma_0 - h(\beta_0 + 4\pi wf)\ ;$$

the rate at which this momentum enters a sphere concentric with the moving sphere is

$$\iint \frac{1}{r}\{g\gamma_0 - h(\beta_0 + 4\pi wf)\}zwd\mathrm{S},$$

where $d\mathrm{S}$ is an element of the surface of the sphere whose radius is r. Integrating this we get

$$-e\beta_0 w\left(1-\frac{w^2}{\mathrm{V}^2}\right)\left(\frac{\mathrm{V}^2}{w^2} - \frac{1}{2}\frac{\mathrm{V}^3}{w^3}\log\frac{(\mathrm{V}+w)}{(\mathrm{V}-w)}\right):$$

this is the increase in momentum per unit time, and therefore measures the force on the space inside the sphere. It will be noticed that this vanishes when $\mathrm{V}=w$: thus a charged sphere, moving with the velocity of light through a magnetic field, will not be acted upon by any force. We may regard the force on the sphere as arising in the following way. Let us suppose that we have a uniform magnetic field parallel to y, then, in consequence of this external magnetic force, the closed tubes of electrostatic induction will be moving about in the field, the positive tubes going one way and the negative ones the opposite. Let us consider what happens when one of these tubes goes through the sphere. If β_1 and β_2 are the values of the magnetic force where it respectively enters and leaves the sphere, the momentum of the tube parallel to x when it enters the sphere is proportional to β_1, and when it leaves the sphere to β_2. Now β_1 and β_2 are different because on one side of the sphere the magnetic force due to the motion of the sphere acts in the same direction as the external magnetic force, but on the other side of the sphere it acts in the opposite direction. Thus the momentum of the tube parallel to x is not the same when it leaves the sphere as before it entered, so that the space inside the sphere must have gained or lost momentum.

The second case we shall consider is that of the magnetic field around a circular current of large radius. Then near a portion of this circuit, which we may consider straight, the tubes of electrostatic induction will be parallel to the circuit and moving radially, the positive tubes (*i.e.* those parallel to the direction of the current) moving inwards, the negative ones outwards ; the positive tubes move in to the conductor and then contract in the manner previously described. The motion of the positive tubes is the same as that given by Prof. Poynting in the Philosophical Transactions, 1884, part 1, p. 350.

In those parts of the field where there is no resultant electromotive intensity there will be as many positive as negative tubes in the field, and when the field is steady the positive ones will flow as fast in one direction as the negative ones in the opposite. Let us consider a region close to the circuit which we may here consider straight; let h be the sum of the strengths of the positive tubes in a unit of the area at right angles to the current, u their radial velocity, h' the sum of the strengths of the negative tubes, u' their radial velocity. Then in unit time the number of positive tubes flowing in plus the number of negative ones flowing out of any cylinder coaxial with the wire, must equal i the intensity of the current. If r is the radius of such a cylinder, the number of positive tubes flowing in in unit time is $hu \times 2\pi r$, the number of negative ones flowing out $h'u' \times 2\pi r$; hence we have

$$(hu + h'u)2\pi r = i.$$

But by equations (5)

$$4\pi(hu + h'u) = \beta,$$

if β is the magnetic force at the surface of the cylinder; as this is at right angles both to the direction of the tubes and also to their velocity, it will be tangential to the cylinder and at right angles to the current. From these equations we find

$$\beta = \frac{2i}{r},$$

the usual expression for the magnetic force close to a current.

The radial momentum inwards $= \Sigma h\beta = \beta\Sigma h$, hence the radial momentum carried across unit area of the cylinder in unit time

$$= \beta\Sigma hu,$$
$$= \frac{\beta^2}{4\pi}.$$

The electromotive intensity due to the motion of the tubes of electrostatic induction is

$$-\Sigma u\beta,$$

and is at right angles both to the magnetic force and the direction of motion of the tubes, so that in the neighbourhood of the current it will be parallel to the current. When the field is in a steady state we have the positive tubes moving with equal velocity but in the opposite direction to the negative ones; in this case $\Sigma u = 0$, and there is no electromotive force due to the motion of the tube. When the current is first started the positive tubes move in before the negative ones begin to move out, so that in this case Σu is positive, and hence the electromotive force is negative, *i. e.* in the opposite direction to the current. When the current is suddenly stopped, the inward flow of tubes is stopped, but the outward one continues for some time; in this case Σu is negative and the electromotive force is therefore positive, *i. e.* in the direction of the original current.

We shall now consider, from the point of view of this theory, experiments such as those made by Rowland on the magnetic effects produced by the rotation of electrified disks. Let us consider a parallel plate-condenser rotating with an angular velocity ω about its axis, which we shall take as the axis of z. Let h be the electric displacement parallel to z; then the components of the magnetic force due to the motion of the tubes whose ends are on the plates of the condenser are given by

$$\alpha = 4\pi\omega x h,$$
$$\beta = 4\pi\omega y h,$$
$$\gamma = 0.$$

These values do not satisfy the solenoidal condition

$$\frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz} = 0.$$

So that inside the condenser we must have, in addition to the motion of those tubes which end on the plates of the condenser, a system of closed tubes in motion such that, while they do not alter the electric displacement, they alter the magnetic force. Thus we must have positive tubes moving in one direction and an equal number of negative ones moving in the opposite. If the motion of these tubes were confined to the inside of the condenser there would be no magnetic force outside the condenser, as the tubes of electrostatic induction outside are then all at rest, and it would follow that in this case there would be no magnetic force outside or inside. As this is not consistent with the results of Rowland's experiments, we shall suppose that these closed tubes pass right through the condenser, their motion being continuous from the inside to the out. The magnetic force due to the motion of these tubes will therefore be continuous as we cross the plates of the condenser. In consequence, however, of the motion of the tubes which have their ends on the plates of the condenser, α and β increase respectively by $4\pi\omega xh$ and $4\pi\omega yh$ as we cross one plate and decrease by the same amount as we cross the other. Thus the conditions by which α, β, γ are determined will be that, except in the plates of the condenser, they are derived from a potential, that they everywhere satisfy the solenoidal condition, that γ is continuous, while α and β increase by $4\pi\omega hx$, $4\pi\omega hy$ respectively as we cross a plate of the condenser. Hence this distribution of magnetic force is exactly what would be produced if we supposed that each moving charge e of electricity produced the same effect as a current ωre, where r is the distance of the charge from the axis of rotation.

Hitherto we have only considered non-magnetic substances. We shall now proceed to discuss the differences which occur when the tubes of electrostatic induction are moving through iron or some other magnetic substance.

If a, b, c are the components of the magnetic induction, α, β, γ those of the magnetic force, the energy in unit volume is

$$\frac{1}{8\pi}(a\alpha + b\beta + c\gamma),$$

and if the magnetism is entirely induced and μ is the magnetic permeability, this equals

$$\frac{\mu}{8\pi}(\alpha^2 + \beta^2 + \gamma^2).$$

From this expression for the kinetic energy we can deduce the components of momentum and electromotive intensity due to a moving tube of electrostatic induction in the same way as we deduced equations (2) and (3). Doing so, we find for U, V, W, the components of momentum, the expressions

$$\begin{aligned} \mathrm{U} &= \mu(g\gamma - h\beta) = gc - hb, \\ \mathrm{V} &= \mu(h\alpha - f\gamma) = ha - fc, \\ \mathrm{W} &= \mu(f\beta - g\alpha) = fb - ga. \end{aligned}$$

And for X, Y, Z, the components of the electromotive force

$$\begin{aligned} \mathrm{X} &= \mu(\bar{w}\beta - \bar{v}\gamma) = \bar{w}b - \bar{v}c, \\ \mathrm{Y} &= \mu(\bar{u}\gamma - \bar{w}\alpha) = \bar{u}c - \bar{w}a, \\ \mathrm{Z} &= \mu(\bar{v}\alpha - \bar{u}\beta) = \bar{v}a - \bar{u}b. \end{aligned}$$

Thus when a tube of electrostatic induction is moving with the same velocity through places of given magnetic force, the momentum it possesses and the electromotive force it produces are proportional to the magnetic permeability of the substance through which it is moving. Hence the expressions for the mechanical force on a conductor conveying a current, and the electromotive force arising from electromagnetic induction, which we have deduced for non-magnetic substances, will be true for magnetic ones if we replace the magnetic force by the magnetic induction.

When the tubes are moving through a field partly occupied by iron, since the inertia of the tubes on the iron is very much greater than in the air, the flow of the tubes through the field will be affected by the iron in much the same way as the flow of a current of electricity would be affected if the air were replaced by a good conductor of electricity and the iron by a bad one.

We shall now consider some problems in which iron is in the field, selecting two-dimensional ones, as in these we avoid as much as possible purely mathematical difficulties.

Let us first take the case when an infinitely long cylinder whose axis is at right angles to the plane of the paper is introduced into a uniform magnetic field, the lines of force being parallel to the plane of the paper and horizontal. The tubes of electrostatic induction which are perpendicular to the plane of the paper were before the introduction of the iron moving vertically. When the bar is introduced they will avoid it, and will spread out so that their paths are like the lines of equipotential surfaces which are given for this case in plate xv. of Maxwell's 'Electricity and Magnetism;' the lines of magnetic force which are at right angles to the lines of flow of the tubes will therefore be the lines of force given in that figure.

Let us now consider the conditions which must be fulfilled at the surface separating iron from air in the general case when the field is not assumed to be uniform. If R is the velocity normal to this surface of a tube of strength h, then $\Sigma h\mathrm{R}$ must be continuous as we cross the surface, otherwise there would be an accumulation or the reverse of tubes at the surface. Now $4\pi\Sigma h\mathrm{R}$ is the tangential magnetic force; thus the tangential magnetic force must be continuous as we cross the surface. Again, the momentum parallel to the surface of a tube will not be altered as it crosses the surface. The momentum of the tube before crossing the surface is $h\rho$, if ρ is the normal magnetic force in air; after crossing the surface it is $h\mu\rho_1$, if ρ_1 is the normal magnetic force in the iron : hence, since the tangential momemtum is constant,

$$\rho = \mu\rho_1.$$

In other words, the normal induction is constant. Hence we have arrived at the usual boundary conditions for the lines of magnetic force.

The intensity of magnetization at the surface is

$$(\rho - \rho_1)/4\pi = \Sigma h(t - t_1),$$

where t and t_1 are the tangential velocities of the tubes in air and iron respectively.

That, from this point of view, magnetization corresponds to a discontinuity in the tangential velocity of the tubes of electrostatic induction.

We can easily prove from this consideration that if the lines of flow of the tubes coincide both inside and outside the cylinder with those due to a solid cylinder moving vertically through an incompressible fluid, the distribution of magnetic force will be that produced by the cylinder, if uniformly magnetized in the horizontal direction.

We have not hitherto determined the velocity with which the tubes of electrostatic induction are moving. We may, however, easily do this when the electromotive intensity is entirely due to the motion of the tubes. For, in this case X, the x component of the electromotive intensity, is given by the equation

$$\begin{aligned} X &= \mu(w\beta - v\gamma), \\ &= 4\pi\mu\{w(wf - uh) - v(ug - vh)\}, \\ &= 4\pi\mu\{(u^2 + v^2 + w^2)f - u(uf + vg + wh)\}, \end{aligned}$$

with similarly expressions for Y and Z.

Since

$$X = \frac{4\pi}{K}f, \quad Y = \frac{4\pi}{K}g, \quad Z = \frac{4\pi}{K}h,$$

we see

$$u^2 + v^2 + w^2 = \frac{1}{\mu K},$$

and $$uf + vg + wh = 0.$$

Hence, in this case, the tubes are moving through the medium at right angles to themselves with the velocity of light.